1. The curve C has equation

$$y = (2x - 3)^5$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(a) the value of w,

(2)

(b) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants.

(5)

a) $y = -32 \quad (-2)^5 = 32 \Rightarrow 2x - 3 = -2$

$2x = 1$

$x = \frac{1}{2} \therefore w = \frac{1}{2}$

b) $\frac{dy}{dx} = 5(2x-3)^4 \times 2 \quad Mt\Big|_{x=\frac{1}{2}} = 5(-2)^4 \times 2 = 160$

$(\frac{1}{2}, -32) \quad y + 32 = 160(x - \frac{1}{2}) \Rightarrow y = 160x - 112$

2.
$$g(x) = e^{x-1} + x - 6$$

(a) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for α.

(b) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

a) $e^{x-1} + x - 6 = 0 \Rightarrow e^{x-1} = 6 - x$

$\Rightarrow x - 1 = \ln(6-x) \Rightarrow x = \ln(6-x) + 1$ #

[$\ln a$, $a > 0$ $\therefore$ $6 - x > 0 \Rightarrow x < 6$]

b) $x_0 = 2$
$x_1 = 2.3863$
$x_2 = 2.2847$
$x_3 = 2.3125$
$\vdots$
$x_n \simeq 2.3065586..$

c) $f(2.3065) = -0.00028 < 0$
$f(2.3075) = 0.00044 > 0$

$\therefore$ by sign change rule, root lies between 2.3065 and 2.3075

$\therefore \alpha = 2.307$ (3dp)

3.

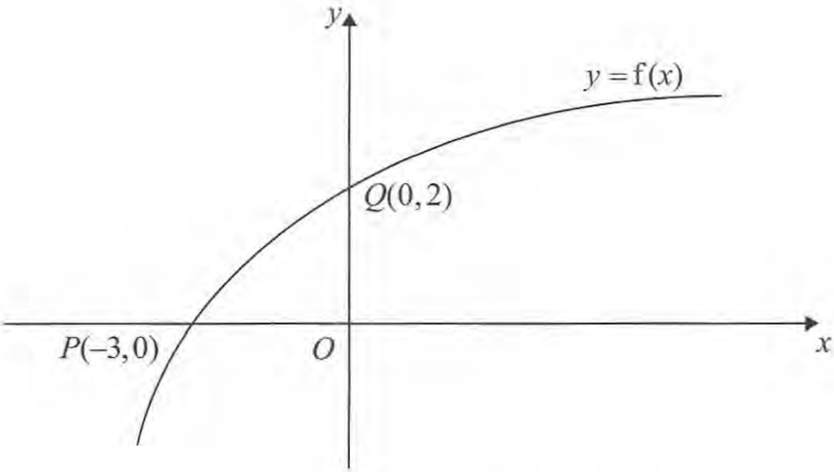


Figure 1

Figure 1 shows part of the curve with equation $y = \mathrm{f}(x), x \in \mathbb{R}$.

The curve passes through the points $Q(0,2)$ and $P(-3,0)$ as shown.

(a) Find the value of $\mathrm{ff}(-3)$.

(2)

On separate diagrams, sketch the curve with equation

(b) $y = \mathrm{f}^{-1}(x)$,

(2)

(c) $y = \mathrm{f}(|x|) - 2$,

(2)

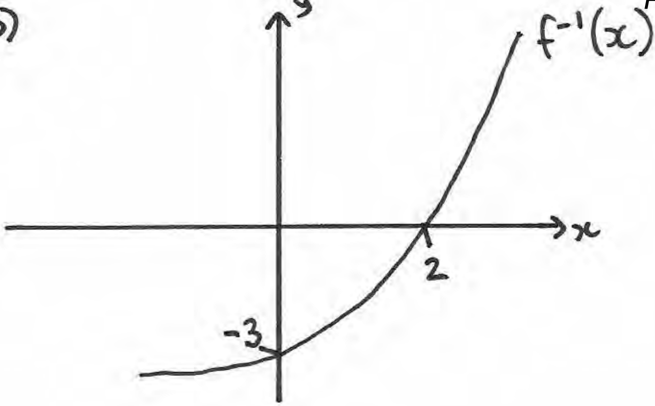
(d) $y = 2\mathrm{f}\left(\frac{1}{2}x\right)$.

(3)

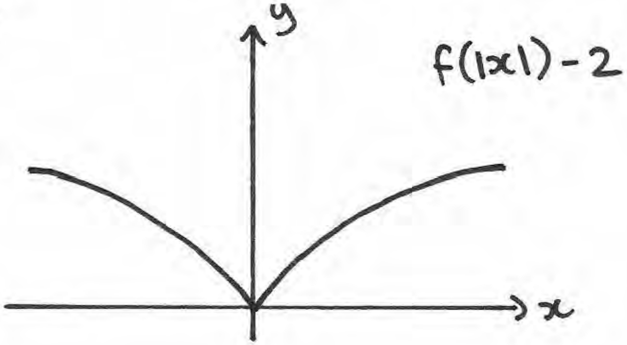
Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

a) $ff(-3) = f(0) = 2$

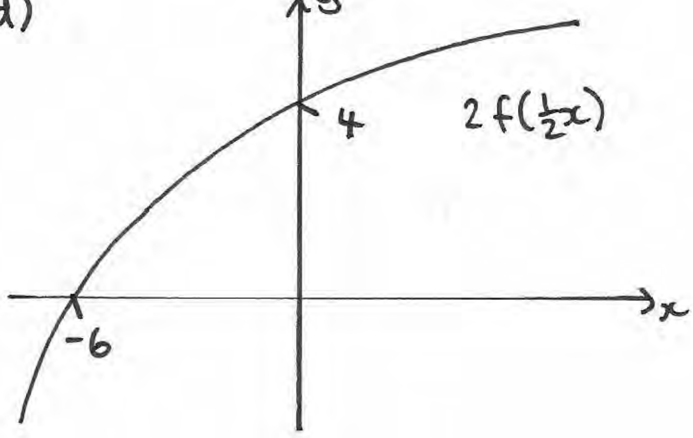
b)



c)



d)



4. (a) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(b)
$$p(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi$$

Calculate

(i) the maximum value of $p(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)

$R\cos(\theta - \alpha) = R\cancel{\cos\theta}\cos\alpha + R\cancel{\sin\theta}\sin\alpha$

$\qquad\qquad\qquad\quad 6\cancel{\cos\theta} \quad + 8\cancel{\sin\theta}$

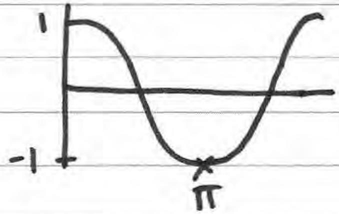
$\frac{R\sin\alpha = 8}{R\cos\alpha = 6} \Rightarrow \tan\alpha = \frac{4}{3} \Rightarrow \alpha = 0.927$

b) $R^2 = 8^2 + 6^2 \Rightarrow R = 10 \quad \therefore \quad 10\cos(\theta - 0.927)$

$p(\theta) = \frac{4}{12 + (10\cos(\theta - 0.927))} \qquad \therefore \text{Max } P(\theta) = \frac{4}{12 - (10)}$

$= \frac{4}{2} = \underline{2}$

ii)



$\therefore$ max occurs when

$\theta - 0.927 = \pi$

$\therefore \underline{\theta = 4.07}$

5. (i) Differentiate with respect to x

(a) $y = x^3 \ln 2x$

(b) $y = (x + \sin 2x)^3$

(6)

Given that $x = \cot y$,

(ii) show that $\frac{dy}{dx} = \frac{-1}{1+x^2}$

(5)

a) $u = x^3 \quad v = \ln 2x$

$u' = 3x^2 \quad v' = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = 3x^2 \ln 2x + x^2$

b) $\frac{dy}{dx} = 3(x + \sin 2x)^2 \times (1 + 2\cos 2x)$

ii) $x = \cot y = \frac{\cos y}{\sin y}$

$u = \cos y \quad v = \sin y$

$u' = -\sin y \quad v' = \cos y$

$\frac{dx}{dy} = \frac{-\sin^2 y - \cos^2 y}{\sin^2 y} = \frac{-(\sin^2 y + \cos^2 y)}{\sin^2 y}$

$\therefore \frac{dy}{dx} = -\sin^2 y$

$x^2 = \cot^2 y$

$\frac{\sin^2 y}{\sin^2 y} + \frac{\cos^2 y}{\sin^2 y} = \frac{1}{\sin^2 y}$

$\Rightarrow 1 + \cot^2 y = \frac{1}{\sin^2 y}$

$\therefore \sin^2 y = \frac{1}{1 + \cot^2 y}$

$\therefore \sin^2 y = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{1}{1+x^2}\right)$

$\therefore \frac{dy}{dx} = \frac{-1}{1+x^2}$ #

ii) easy way $x = \cot y$ $\frac{dx}{dy} = -\operatorname{cosec}^2 y$

$\therefore \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$

$\operatorname{cosec}^2 y = 1 + \cot^2$

$\Rightarrow \operatorname{cosec}^2 y = 1 + x^2$

$\therefore \frac{dy}{dx} = \frac{-1}{1+x^2}$ #

6. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2$$

You must show each stage of your working.

(5)

(ii) (a) Show that $\cos 2\theta + \sin\theta = 1$ may be written in the form

$$k\sin^2\theta - \sin\theta = 0, \text{ stating the value of } k.$$

(2)

(b) Hence solve, for $0 \leqslant \theta < 360°$, the equation

$$\cos 2\theta + \sin\theta = 1$$

(4)

i) $= \sin^2 22.5 + 2\sin 22.5\cos 22.5 + \cos^2 22.5$

$= (\sin^2 22.5 + \cos^2 22.5) + \sin 45$

$= 1 + \frac{\sqrt{2}}{2}$

ii) $\cos 2\theta + \sin\theta = 1$

$\Rightarrow (1 - 2\sin^2\theta) + \sin\theta = 1$

$\Rightarrow \cancel{1} - 2\sin^2\theta + \sin\theta = \cancel{1} \Rightarrow 2\sin^2\theta - \sin\theta = 0$

$\therefore \underline{k = 2}$

b) $(2\sin\theta - 1)\sin\theta = 0$

$\therefore \sin\theta = 0 \Rightarrow \theta = 0, 180$

$\therefore \sin\theta = \frac{1}{2} \Rightarrow \theta = 30, 150$

7.
$$h(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \qquad x \geqslant 0$$

(a) Show that $h(x) = \dfrac{2x}{x^2+5}$ **(4)**

(b) Hence, or otherwise, find $h'(x)$ in its simplest form. **(3)**

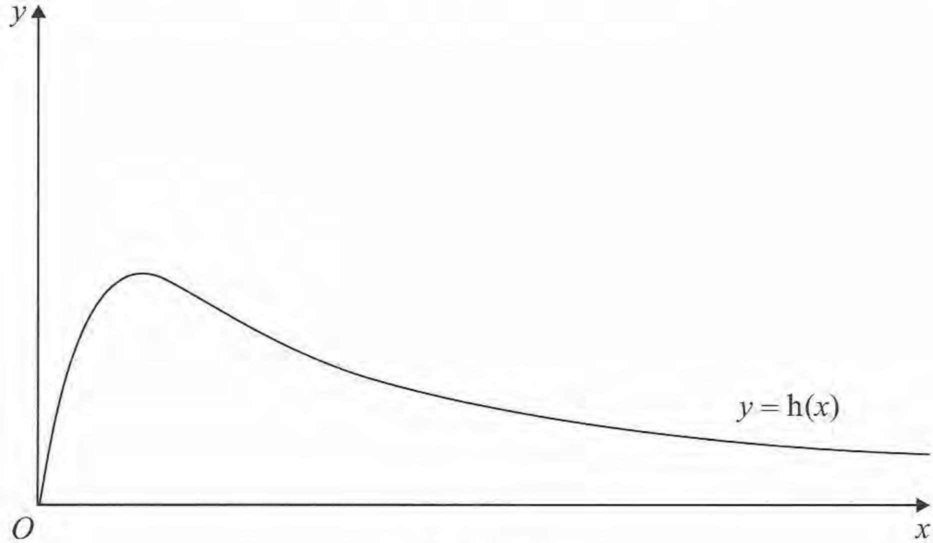


Figure 2

Figure 2 shows a graph of the curve with equation $y = h(x)$.

(c) Calculate the range of $h(x)$. **(5)**

a) $h(x) = \dfrac{2(x^2+5)+4(x+2)-18}{(x+2)(x^2+5)}$

$\Rightarrow \dfrac{2x^2+10+4x+8-18}{(x+2)(x^2+5)} = \dfrac{2x^2+4x}{(x+2)(x^2+5)}$

$\Rightarrow \dfrac{2x\cancel{(x+2)}}{\cancel{(x+2)}(x^2+5)} = \dfrac{2x}{x^2+5}$ #

b) $u = 2x \quad v = x^2+5$
$u' = 2 \quad v' = 2x$

$h'(x) = \dfrac{2x^2+10-4x^2}{(x^2+5)^2}$

$\therefore h'(x) = \dfrac{-2x^2+10}{(x^2+5)^2}$

c) $y \geq 0$ and $y \leq$ max point at TP.

TP when $h'(x) = 0$

$\therefore \quad 2x^2 = 10 \Rightarrow x^2 = 5 \Rightarrow x = \sqrt{5}$

$y = \dfrac{2\sqrt{5}}{5+5} = \dfrac{1}{5}\sqrt{5}$

$\therefore \quad 0 \leq y \leq \dfrac{\sqrt{5}}{5}$

8. The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$$

where V is the value of the car in pounds (£) and t is the age in years.

(a) Find the value of the car when $t = 0$ **(1)**

(b) Calculate the exact value of t when $V = 9500$ **(4)**

(c) Find the rate at which the value of the car is decreasing at the instant when $t = 8$. Give your answer in pounds per year to the nearest pound. **(4)**

a) $t=0 \Rightarrow V = 17000 + 2000 + 500 = \underline{19500}$

b) $17000e^{-0.25t} + 2000e^{-0.5t} + 500 = 9500$

$\Rightarrow 2000(e^{-0.25t})^2 + 17000(e^{-0.25t}) - 9000 = 0$

$\therefore 2000x^2 + 17000x - 9000 = 0 \quad x = e^{-0.25t}$

$\Rightarrow 2x^2 + 17x - 9 = 0$

$(2x - 1)(x + 9) = 0 \quad x = \frac{1}{2}, x = -9$

$\therefore e^{-0.25t} = \frac{1}{2} \Rightarrow -0.25t = \ln(\frac{1}{2}) = -\ln 2$

$\Rightarrow \frac{1}{4}t = +\ln 2 \quad \therefore t = \underline{4\ln 2}$

and $e^{-0.25t} = -9 \Rightarrow -\frac{1}{4}t = \ln(-9)$

$\therefore$ no solution

$\therefore \underline{t = 4\ln 2}$

c) $\frac{dV}{dt} = -4250e^{-0.25t} - 1000e^{-0.5t}$

$t=8 \quad \frac{dV}{dt} = -593.49...$ decreasing at a rate of £593